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Bekenstein bounds in de Sitter and flat space

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ABSTRACT: The D-bound on the entropy of matter systems in de Sitter space is shown to be closely related to the Bekenstein bound, which applies in a flat background. This holds in arbitrary dimensions if the Bekenstein bound is calibrated by a classical Geroch process. We discuss the relation of these bounds to the more general bound on the entropy to area ratio. We find that black holes do not saturate the Bekenstein bound in dimensions greater than four.

KEYWORDS: M-Theory, Black Holes.

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1. Introduction

Bekenstein [1] has argued that isolated, stable thermodynamic systems in asymptotically flat space satisfy the universal entropy bound

$$S_{\mathrm{m}} \leq \frac{2\pi RE}{\hbar c}\,. \tag{1.1}$$

Here R is the radius of a sphere circumscribing a system of total energy E.

The Bekenstein bound has been supported in two independent ways, empirical and logical:

1. A strong case has been made that all physically reasonable, weakly gravitating matter systems satisfy the bound [2, 3]; some come within an order of magnitude of saturation. The bound is exactly saturated by Schwarzschild black holes, for which $S = \pi R^2 c^3/G\hbar$ and $R = 2GE/c^4$. (Henceforth, $c = \hbar = G = 1$.) No realistic matter system exceeding the bound is known. This empirical evidence suggests that the bound is both true and as tight as possible.

2. For weakly gravitating systems, the bound is claimed [1] to follow from the generalized second law of thermodynamics [4]–[6] (for short, 'second law'). Namely, there exists a gedankenexperiment, the *Geroch process*, by which the system is

deposited into a large black hole such that the black hole horizon area grows by no more than $\Delta A = 8\pi RE$. This process increases the black hole entropy by $\Delta A/4$, while the matter entropy, $S_{\rm m}$, is lost. By the second law, the total entropy must not decrease: $\Delta A/4 - S_{\rm m} \geq 0$.

The empirical argument has been called into question by claims that certain systems violate the Bekenstein bound; see, e.g. [7] and references therein. Many of these counter-examples, however, fail to include the whole gravitating mass of the system in E. Others involve questionable matter content, e.g. a very large number of species. If Bekenstein's bound is taken to apply only to complete systems that can actually be constructed in nature, it has not been ruled out [8, 9].

The logical argument is also controversial. The question is whether or not the Bekenstein bound can be derived from the second law via the Geroch process once quantum effects are included [10]–[12]. This is a complicated problem, and a consensus on its proper analysis has yet to be reached [13, 9]; we will not address the question here.

In this paper we adopt the working hypothesis that a correct entropy bound is obtained from the classical analysis of the Geroch process. This will permit us to examine Bekenstein's result from a different viewpoint, focussing on aspects of the following questions: What is the bound's actual scope? And what is its relation to the holographic entropy bound, $S \leq A/4$?

On a generic space-time background, neither the energy nor the radius of a system can be satisfactorily defined. This poses a fundamental restriction on the generality of any bound of the form RE. By contrast, the holographic entropy bound, $S \leq A/4$, can be successfully formulated for arbitrary surfaces in general space-times [14]. Yet the Bekenstein bound is much stronger than the area bound for weakly gravitating systems ($E \ll R$) in flat space. It is of interest to understand whether the holographic bound can be tightened under any other conditions. For example, the proposal of ref. [15] is tighter than the holographic bound but weaker than Bekenstein's. It has a wide range of validity, though it does not appear to apply to some black hole interiors.

Here we ask, in particular, if a Bekenstein-type bound can hold in space-times that are not asymptotically flat. In order to separate this question from the problem of strong gravitational dynamics due to the system's self-gravity, we consider weakly gravitating systems in a curved vacuum background generated by a cosmological constant Λ.

Asymptotically de Sitter space-times ($\Lambda > 0$) are of special interest because there is a cosmological horizon surrounding the observer. The generalized second law must hold for matter systems crossing this horizon, just as for a black hole horizon [16, 17]. Thus one obtains a bound on the entropy of matter systems within the cosmological horizon, the 'D-bound' [18]. We review this argument in section 2. The D-bound depends on the cosmological constant and the initial horizon area. This form is useful for at least one application [18], but it obscures the D-bound's relationship to the flat space Bekenstein bound.

In section 3 we evaluate the D-bound in the dilute limit, i.e. for weakly gravitating, approximately spherical systems that extend as far as the cosmological horizon. We find that the D-bound takes the same form as the Bekenstein bound, if the latter is expressed in terms of the 'gravitational radius' rather than the energy of the system. (The gravitational radius can be generally defined, while energy is not meaningful in de Sitter space.) Although a special limit is taken, the agreement is non-trivial, because the background geometry differs significantly from flat space. Thus, one may regard the D-bound, in its general form, as a de Sitter space equivalent of the flat-space Bekenstein bound.

In section 4 we generalize the derivation of the Bekenstein bound from the Geroch process to space-time dimension $D > 4$, both for asymptotically de Sitter spaces (using the cosmological horizon) and for asymptotically flat space (using a black hole). This serves two purposes. We confirm that the agreement between the de Sitter and flat space cases continues to be exact, including the numerical prefactor, in arbitrary space-time dimensions. But we also find a puzzling result (section 5). Schwarzschild black holes, which saturate the Bekenstein bound in $D = 4$, fall short of the bound in higher dimensions, by a factor of order one. Put differently, the Bekenstein bound does not imply the holographic bound for spherical systems in $D > 4$.

2. The D-bound on matter entropy in de Sitter space

2.1 Derivation

de Sitter space is the maximally symmetric positively curved space-time. It is a vacuum solution to Einstein's equations with a positive cosmological constant, Λ. The radius of curvature is given by

$$r_0 = \sqrt{\frac{3}{\Lambda}}\,. \tag{2.1}$$

An observer at $r = 0$ is surrounded by a cosmological horizon at $r = r_0$. This is manifest in the static coordinate system:

$$ds^2 = -V(r)\, dt^2 + \frac{1}{V(r)} dr^2 + r^2 d\Omega_2^2\,, \tag{2.2}$$

where

$$V(r) = 1 - \frac{r^2}{r_0^2}\,. \tag{2.3}$$

These coordinates cover only part of the space-time, namely the interior of a cavity bounded by $r = r_0$, just as a static coordinate system will cover only the exterior of a Schwarzschild black hole in flat space.

An object held at a fixed distance from the observer is redshifted; the red-shift diverges near the horizon. If released, the object will move towards the horizon. If it crosses the horizon, it cannot be retrieved. Thus, the cosmological horizon acts like a black hole 'surrounding' the observer. Note that the symmetry of the space-time implies that any point can be called $r = 0$, so the location of the cosmological horizon is observer-dependent.

The analogy between the de Sitter horizon and a black hole persists at the semi-classical level [16]. Consider a process whereby a matter system is dropped into a black hole. Because the matter entropy is lost, one may be concerned that the second law of thermodynamics is violated. However, the black hole becomes larger in the process. If the horizon is assigned an entropy equal to a quarter of its surface area, black hole entropy grows at least enough to compensate for the lost matter entropy, and so a 'generalized second law' [4]–[6] holds for the total entropy. Similarly, because matter entropy can be lost when it crosses the cosmological horizon of de Sitter space, the horizon surface must also be assigned a Bekenstein-Hawking entropy:

$$S_0 = \frac{1}{4} A_0 \,, \tag{2.4}$$

where

$$A_0 = 4\pi r_0^2 = \frac{12\pi}{\Lambda} \tag{2.5}$$

is the area of the cosmological horizon.

A large class of space-times with $\Lambda > 0$ are asymptotically de Sitter in the future. This means that there exists a future region far from black holes or other matter in which the space-time looks locally like empty de Sitter space. An observer whose world-line moves into this region will find that ordinary matter falls away from the observer, towards the cosmological horizon. In the end, only vacuum energy remains, and the solution is locally described by eq. (2.2).

Consider a matter system within the apparent cosmological horizon of an observer, in a universe that is asymptotically de Sitter in the future. Let the observer move relative to the matter system, into the asymptotic region. The observer will witness a thermodynamic process by which the matter system is dropped across the cosmological horizon, while the space-time converts to empty de Sitter space. The entropy of the final state is $S_0 = A_0/4$. The entropy of the initial state is the sum of the matter entropy, S_{m}, and the Bekenstein-Hawking entropy, which is given by a quarter of the area of the apparent cosmological horizon:

$$S = S_{\mathrm{m}} + \frac{1}{4} A_{\mathrm{c}} \,. \tag{2.6}$$

By the generalized second law, $S \leq S_0$. This yields a bound[1]on the matter

[1]The reader interested in the controversial effects of quantum buoyancy on the Geroch pro-

entropy in terms of the change of the horizon area:

$$S_{\rm m} \le \frac{1}{4}\left(A_0 - A_{\rm c}\right). \tag{2.7}$$

This will be called the *D-bound* on matter systems in asymptotically de Sitter space. The D-bound vanishes for empty de Sitter space ($A_{\rm c} = A_0$), where indeed there is no matter present. Since $S_{\rm m} \ge 0$, the D-bound implies in particular that $A_{\rm c} \le A_0$. That is, a cosmological horizon enclosing matter must have smaller area than the horizon of empty de Sitter space.

2.2 Example

As a simple example, consider a black hole in (asymptotically) de Sitter space. Then the 'matter entropy', $S_{\rm m}$, is the Bekenstein-Hawking entropy of the black hole. We will verify that it satisfies the D-bound, and in particular that $A_{\rm c}$, the area of the cosmological horizon surrounding the black hole, is less than A_0.

The family of Schwarzschild-de Sitter solutions can be written in the form

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_2^2\,, \tag{2.8}$$

where

$$V(r) = 1 - \frac{2M}{r} - \frac{r^2}{r_0^2}\,. \tag{2.9}$$

The mass parameter M lies in the range $[0, 1/3\sqrt{\Lambda}]$. A black hole in de Sitter space may not be arbitrarily large because it must fit within the cosmological horizon.

The locations of the black hole and cosmological horizons are given by the positive roots, $r_{\rm b}$ and $r_{\rm c}$, of the cubic equation $V(r) = 0$. Their values depend on the mass parameter M. The choice $M = 0$ corresponds to empty de Sitter space. In this case V has only one positive root, $r_{\rm c} = r_0$, which is the radius of the cosmological horizon. For $M > 0$, there will be a second root corresponding to a black hole horizon ($r_{\rm b} \approx 2M$ for small M). As the parameter M is increased, one easily finds that the black hole radius r_b increases and the cosmological radius $r_{\rm c}$ *decreases* monotonically. They become equal, $r_{\rm b} = r_{\rm c} = \sqrt{1/\Lambda}$, for the maximal value of M.

Recall that the cosmological horizon in empty de Sitter space has area $A_0 = 4\pi r_0^2$. With a matter system present, the cosmological horizon has area $A_{\rm c} = 4\pi r_{\rm c}^2$. Since $r_{\rm c}(M = 0) = r_0$ and $r_{\rm c}(M > 0)$ decreases monotonically, we have verified that $A_{\rm c} \le A_0$ for all values in the range of M, with equality only for $M = 0$. That is, the cosmological horizon around a Schwarzschild-de Sitter black hole is smaller than the horizon of empty de Sitter space.

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cess [13, 9] will note that there is little room for such effects in our gedankenexperiment in de Sitter space. There is no need for the slow lowering of a system to large redshifts. At most, a small correction to the restriction on the size of the system should be made, to ensure that it does not approach the immediate vicinity of the cosmological horizon, where redshifts are high.

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We may now verify the D-bound, which states in this example that the entropy of the black hole, $A_{\rm b}/4 = \pi r_{\rm b}^2$, is bounded by $\pi(r_0^2 - r_c^2)$. Equivalently, one may check that the entropy of Schwarzschild-de Sitter space,

$$S(M) = \pi(r_{\rm b}^2 + r_{\rm c}^2)\,, \tag{2.10}$$

is less than the entropy of empty de Sitter space:

$$S(0) = S_0 = \pi r_0^2\,. \tag{2.11}$$

By solving the cubic equation $V = 0$ for its positive roots one finds that $S(M)$ is a monotonically decreasing function of the mass parameter, as required. More precisely,

$$S = \pi r_0^2 \left(1 - \frac{2M}{r_0}\right) + O(M^2) \tag{2.12}$$

for small M; and $S(M) = \frac{2}{3}S_0$ for the maximal black hole. Thus, in the example of Schwarzschild-de Sitter black holes, the D-bound is upheld with room to spare.

For general black hole configurations with total area A_{rmB} one can in fact prove the inequality [19]

$$A_0 - A_{\rm B} - A_{\rm c} \geq \sqrt{A_{\rm B} A_{\rm c}}\,. \tag{2.13}$$

Since the right hand size is obviously non-negative, this provides us with a more general check of the D-bound. The inequality is saturated by the Schwarzschild-de Sitter solution. The useful identity

$$r_0^2 = r_{\rm b}^2 + r_{\rm c}^2 + r_{\rm b} r_{\rm c} \tag{2.14}$$

is obtained by setting $V(r_{\rm b}) = V(r_{\rm c}) = 0$. The mass parameter, M, can be eliminated in favour of the black hole radius, $r_{\rm b}$:

$$2M = r_{\rm b} \left(1 - \frac{r_{\rm b}^2}{r_0^2}\right). \tag{2.15}$$

It should not come as a surprise that small black holes ($0 < M \ll \Lambda^{-1/2}$ fall short of saturating the D-bound. (So do large black holes but the explanation below will not apply.) Suppose we hold fixed the mass and the spherical volume accessible to a system in asymptotically flat space. One may ask how the entropy can be maximized. This will not necessarily be achieved by condensing the mass into a small black hole. Often, a larger number of modes can be accessed by diluting the mass over the entire volume. This result is also supported by Bekenstein's bound, eq. (2.2). The bound is not maximized by setting $R = E$, but rather by letting R be as large as possible at fixed E. One would not expect the presence of the cosmological constant to change this behaviour qualitatively. In the present context, we may consider the cosmological horizon to be fixing the maximal volume. Small Schwarzschild-de Sitter spaces do not make use of this volume. Hence, dilute systems with the same cosmological horizon might have higher entropy. It is a pleasing result, therefore, that small black holes do not saturate the D-bound.

3. Relation between D-bound and Bekenstein bound

The flat space Bekenstein bound, eq. (1.1), makes no reference to the auxiliary black hole employed in the Geroch process. The area increase is expressed only in terms of characteristics of the matter system: its energy and radius. Can the D-bound, by analogy, be formulated without reference to the cosmological constant? In this section we will find such an expression, for a limited class of systems.

In de Sitter space, the energy of the system is not well-defined, for lack of a suitable asymptotic region. For a spherical system, however, Birkhoff's theorem implies that there exists some Schwarzschild-de Sitter solution that has the same metric at large radii; in particular, it has the same cosmological horizon radius r_c. We call this black hole the system's *equivalent black hole*, and its radius the system's *gravitational radius*, $r_{\rm g}$.

In flat space, the gravitational radius would just be twice the energy, and one may express the Bekenstein bound in terms of either quantity. In asymptotically de Sitter space, $r_{\rm g}$ can still be defined while the energy cannot. Moreover, any one of the quantities $(r_{\rm g}, r_{\rm c}, r_0)$ is determined by the other two. Thus, if one characterizes the system by $r_{\rm g}$ and $r_{\rm c}$, one can eliminate A_0 in eq. (2.7).

With $r_{\rm b} = r_{\rm g}$, substitution of Eq. (2.14) into the D-bound, Eq. (2.7), yields

$$S_{\rm m} \leq \pi r_{\rm g} r_{\rm c} \left(1 + \frac{r_{\rm g}}{r_{\rm c}}\right) . \tag{3.1}$$

This expression is particularly interesting in the limit of small equivalent black holes, or

$$r_{\rm g} \ll r_c . \tag{3.2}$$

This approximation corresponds to 'light' matter systems, for which the cosmological horizon area is nearly A_0. To leading order in $r_{\rm g}$, the D-bound thus takes the form

$$S_{\rm m} \leq \pi r_{\rm g} r_{\rm c} . \tag{3.3}$$

In words, the entropy of a spherical system in de Sitter space cannot be larger than π times the product of its gravitational radius and the radius of the cosmological horizon.

Compare this with the Bekenstein bound in flat space, expressed in terms of the gravitational radius $r_{\rm g} = 2m$:

$$S_{\rm m} \leq \pi r_{\rm g} R , \tag{3.4}$$

where R is the radius of a sphere circumscribing the system. But in de Sitter space, a stable system cannot be larger than $R = r_c$. Thus, for dilute spherical systems, the D-bound coincides with Bekenstein's flat space bound. In this limit, the agreement is exact, including the numerical factor π.

The geometry occupied by a dilute system extending to the cosmological horizon of de Sitter space deviates strongly from flat space, so the agreement of the bounds is non-trivial. The simplicity of the de Sitter gedankenexperiment in comparison to the more complex Geroch process makes the coincidence particularly striking.

It is interesting to compare the D-bound, eq. (2.7), to the holographic entropy bound applied to the cosmological horizon area,

$$S \leq \frac{A_{\rm c}}{4}\,. \tag{3.5}$$

In asymptotically de Sitter space, $A_{\rm c}$ ranges between $A_0/3$ (for the maximal Schwarzschild-de Sitter solution) and A_0 (for empty de Sitter space). If $A_{\rm c} > A_0/2$, the D-bound will be tighter than the holographic bound, which states that $S_{\rm m} \leq \frac{1}{4}A_{\rm c}$. For $A_{\rm c} < A_0/2$ the holographic bound is tighter; in fact it is saturated by a maximal black hole, for which $A_{\rm b} = A_{\rm c} = A_0/3$.

The combination of both bounds, $\mathrm{Min}\{\frac{A_{\rm c}}{4}, \frac{A_0 - A_{\rm c}}{4}\}$, is not differentiable at $A_{\rm c} = A_0/2$. If the bounds have a common origin, one would expect that there exists a smooth interpolation which is at least as tight as the minimum of the two bounds. It must coincide where saturating examples are known, i.e. at the extremes $A_{\rm c} = A_0$ and $A_{\rm c} = A_{\rm b}$. Note that eq. (3.3), which was derived only for small $r_{\rm g}$, satisfies these properties when extended to the full range $0 \leq 4\pi r_g^2 \leq A_0/3$.

The following section has two purposes. It will verify that the agreement between Bekenstein bound and D-bound persists in space-times of dimension greater than four. Also, it will prepare the ground for section 5, where we note that neither bound is saturated by black holes for general D.

4. Bounds in higher dimensions

4.1 D-bound in $D > 4$

In section 2 the D-bound was obtained in the general form

$$S_{\rm m} \leq \frac{1}{4}\left(A_0 - A_{\rm c}\right), \tag{4.1}$$

valid for all systems within the cosmological horizon of de Sitter space. In section 3 this expression was converted to the special form

$$S_{\rm m} \leq \pi r_{\rm g} r_{\rm c} \qquad (D = 4)\,, \tag{4.2}$$

valid only for light spherical systems. For systems that extend all the way to the cosmological horizon, the D-bound and the Bekenstein bound were thus found to agree exactly.

Equation (4.1) is manifestly independent of the space-time dimension $D = n+1$, as one may take $A_{\rm c}$ and A_0 to be the $n-1$ dimensional areas of the cosmological horizon. However, the dimension does enter in the derivation of eq. (4.2), which we now extend to $D > 4$.

Assuming spherical symmetry, the D-bound can be written as

$$S_{\rm m} \le \frac{1}{4}\mathcal{A}_{n-1}\left(r_0^{n-1} - r_{\rm c}^{n-1}\right), \tag{4.3}$$

where $\mathcal{A}_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of a unit $n-1$ sphere. We wish to eliminate r_0 in order to avoid reference to the cosmological constant. In analogy to the previous section, one may express the D-bound in terms of the matter system's gravitational radius, $r_{\rm g}$, and its maximal size, $r_{\rm c}$.

The Schwarzschild-de Sitter metric is given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_{n-1}^2\,, \tag{4.4}$$

where

$$V(r) = 1 - \left(1 - \frac{r_{\rm g}^2}{r_0^2}\right)\left(\frac{r_{\rm g}}{r}\right)^{n-2} - \frac{r^2}{r_0^2}\,. \tag{4.5}$$

The physically meaningless mass parameter has been replaced by the black hole radius, $r_{\rm b} = r_{\rm g}$. The cosmological horizon is the larger positive root of $V(r)$. $V(r_{\rm c}) = 0$ implies

$$r_0^2\left[1 - \left(\frac{r_{\rm g}}{r_{\rm c}}\right)^{n-2}\right] = r_{\rm c}^2\left[1 - \left(\frac{r_{\rm g}}{r_{\rm c}}\right)^{n}\right]. \tag{4.6}$$

To leading order in $r_{\rm g}/r_{\rm c}$, one finds

$$r_0^{n-1} = r_{\rm c}^{n-1}\left[1 + \frac{n-1}{2}\left(\frac{r_{\rm g}}{r_{\rm c}}\right)^{n-2}\right]. \tag{4.7}$$

With eq. (4.3), it follows that the D-bound evaluates to

$$S_{\rm m} \le \frac{n-1}{8}\mathcal{A}_{n-1} r_{\rm g}^{n-2} r_{\rm c} \tag{4.8}$$

for a spherical system with $r_{\rm g} \ll r_{\rm c}$, in an asymptotically de Sitter space-time with $D = n+1$ space-time dimensions.

4.2 Geroch process and Bekenstein bound in $D > 4$

In order to generalize the Bekenstein bound to more than four dimensions, we analyze the Geroch process classically for $D = n+1$.

Consider a weakly gravitating stable thermodynamic system of total energy E. Let R be the radius of the smallest $n-1$ sphere circumscribing the system. To obtain an entropy bound, one may move the system from infinity into a Schwarzschild black hole whose radius, b, is much larger than R but otherwise arbitrary. One would like to add as little energy as possible to the black hole, so as to minimize the increase of the black hole's horizon area and thus to optimize the tightness of the entropy bound. Therefore, the strategy is to extract work from the system by lowering it slowly until it is just outside the black hole horizon, before one finally drops it in.

The mass added to the black hole is given by the energy E of the system, red-shifted according to the position of the center of mass at the drop-off point, at which the circumscribing sphere almost touches the horizon. Within its circumscribing sphere, one may orient the system so that its center of mass is 'down', i.e. on the side of the black hole. Thus the center of mass can be brought to within a proper distance R from the horizon, while all parts of the system remain outside the horizon. Hence, one must calculate the redshift factor at radial proper distance R from the horizon.

The Schwarzschild metric is given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_{n-1}^2\,, \tag{4.9}$$

where

$$V(r) = 1 - \left(\frac{b}{r}\right)^{n-2} \equiv [\chi(r)]^2 \tag{4.10}$$

defines the redshift factor, χ. The black hole radius is related to the mass at infinity, M, by

$$b^{n-2} = \frac{16\pi M}{(n-1)\mathcal{A}_{n-1}}\,. \tag{4.11}$$

The black hole has horizon area

$$A = \mathcal{A}_{n-1} b^{n-1} \tag{4.12}$$

and entropy

$$S_{\mathrm{bh}} = \frac{A}{4}\,. \tag{4.13}$$

Let c be the radial coordinate distance from the horizon:

$$c = r - b\,. \tag{4.14}$$

Near the horizon, the redshift factor is given by

$$\chi^2(c) = (n-2)\frac{c}{b}\,, \tag{4.15}$$

to leading order in c/b. The proper distance l is related to the coordinate distance c as follows:

$$l(c) = \int_0^c \frac{dc}{\chi(c)} = 2\left(\frac{bc}{n-2}\right)^{1/2}\,. \tag{4.16}$$

Hence,

$$\chi(l) = \frac{n-2}{2b}\,l\,. \tag{4.17}$$

The mass added to the black hole is

$$\delta M \leq E\,\chi(l)\Big|_R = \frac{n-2}{2b}\,ER\,. \tag{4.18}$$

By eqs. (4.11), (4.12), and (4.13), the black hole entropy increases by

$$\delta S_{\text{bh}} = \frac{dS_{\text{bh}}}{dM}\,\delta M \leq 2\pi E R. \tag{4.19}$$

By the generalized second law, this increase must at least compensate for the lost matter entropy: $\delta S_{\text{bh}} - S_{\text{m}} \geq 0$. Hence,

$$S_{\text{m}} \leq 2\pi E R\,. \tag{4.20}$$

Expressed in terms of energy, the Bekenstein bound is thus independent of the dimension.

Alternatively, one may characterize the system by its gravitational radius,

$$r_{\text{g}}^{n-2} = \frac{16\pi E}{(n-1)\mathcal{A}_{n-1}}\,. \tag{4.21}$$

This yields an equivalent form of the Bekenstein bound,

$$S_{\text{m}} \leq \frac{n-1}{8}\mathcal{A}_{n-1} r_{\text{g}}^{n-2} R\,, \tag{4.22}$$

which does depend on the space-time dimension, but is manifestly independent of the mass normalization chosen in eq. (4.11). Comparison with the higher-dimensional D-bound, eq. (4.8), shows that the agreement found in section 3 for $D = 4$ persists in higher dimensions.

5. Black holes and the Bekenstein bound

The entropy of a Schwarzschild black hole of radius R, in $D = n+1$ space-time dimensions, is given by a quarter of its area in Planck units:

$$S_{\text{bh}} = \frac{1}{4}\mathcal{A}_{n-1} R^{n-1}\,. \tag{5.1}$$

By definition, a black hole's radius is equal to its gravitational radius: $R = r_{\text{g}}$. Thus, comparison with eq. (4.22) shows that a black hole satisfies the Bekenstein bound for all $D \geq 4$. However, it does not saturate the bound except in $D = 4$, missing by a factor of $\frac{D-2}{2}$. Similarly, maximal Schwarzschild-de Sitter black holes do not saturate the D-bound, in the form of eq. (4.8), for $D > 4$. (We do not consider $D < 4$, as there are no regular black hole solutions.)

The derivation of the Bekenstein bound from a Geroch process is valid only for weakly gravitating systems, whose back-reaction on the ambient geometry is negligible. From this perspective, one had no right to expect that black holes would satisfy the bound, let alone that they would saturate it. Nevertheless, the failure of black holes to saturate the Bekenstein bound in $D > 4$ is puzzling for a number of reasons.

Black holes are the most condensed objects with a static external geometry. In $D = 4$, they precisely saturate the Bekenstein bound. This has been viewed as evidence that the bound may apply to a wide range of systems with intermediate self-gravity, up to and including black holes, and that it is the tightest such bound possible. Our result means that the latter conclusion cannot be drawn in $D > 4$.

Using only gravitational stability, i.e. $m \leq R/2$ or $r_{\mathrm{g}} \leq R$, eq. (1.1) becomes $S \leq \pi R^2 = A/4$. For spherically symmetric systems in $D = 4$, the Bekenstein bound thus implies the holographic bound. From eq. (4.22) it is clear that the holographic bound does not follow in the same way for $D > 4$. Instead, one obtains $S \leq \frac{D-2}{8} A$, which is weaker.

The holographic bound can be inferred directly from the second law in arbitrary D by way of a different gedankenexperiment [20] (see also refs. [9, 21, 22]). Indeed, the holographic principle [23, 20] has come to be viewed as fundamental to quantum gravity [24, 25]. In its covariant formulation [26], it implies both the holographic entropy bound [14] and the generalized second law [27] (see ref. [28] for a review). Thus one is faced with two apparently independent entropy bounds in $D > 4$, one of which is tighter for $r_{\mathrm{g}} \ll R$, while the other is tighter for $r_{\mathrm{g}} \to R$.

Unless one assumes that the Bekenstein bound is either invalid or unrelated to the holographic bound, there are two possibilities of how this tension might be resolved. It may be that the Geroch process does not yield the tightest bound possible in $D > 4$, and that instead the stronger inequality

$$S_{\mathrm{m}} \leq \frac{1}{4} \mathcal{A}_{n-1} r_{\mathrm{g}}^{n-2} R \tag{5.2}$$

holds. A second possibility is that eq. (4.22) is already optimally tight for $r_{\mathrm{g}} \ll R$, but that a bound exists that interpolates smoothly between regimes of weak and intermediate gravity, limiting to the holographic bound for $r_{\mathrm{g}} \to R$.

It may turn out that the question can be answered by a more sophisticated gedankenexperiment that infers a suitable bound from the second law, modulo the controversial issue of quantum buoyancy. The covariant entropy bound [14] is stronger than the second law [27]; one may also attempt to derive a Bekenstein-type bound directly from it.

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